Instability of liquid film flowing down a linearly heated plate*

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Abstract The full-scale linear stability equations for a liquid film flowing down a linearly heated inclined plate are derived from the N-S equations, and the stability characters of both temporal and spatial modes are computed by using Chebyshev spectral collocation method. The effects of Weber numbers and Marangoni numbers on growth rate, marginal curve, critical Reynolds number, etc. are investigated. An explicit dispersion relation under long-wave approximation, which is in exact agreement with Miladinova's one, is obtained and the limits of long-wave approximation are discussed.

Keywords: falling film, Marangoni instability, surface wave instability.

A liquid film flowing down an inclined plate (the inclined angle is not small) is susceptible to surface wave instability. The surface wave instability has been studied intensively since the early work by Yih[1] and Benjamin[2]. Their linear stability analyses showed that the flow is stable beyond a cutoff wavenumber, which means long waves are unstable. Weakly nonlinear analyses performed by Lin[3], Gjevik[4] and Pumir et al.[5] showed that there exist supercritical and subcritical instabilities. Recently more attention is given to falling films on a non-uniformly heated plate due to flow control and industrial practice. Kalitzova et al.[6] first conducted the linear long-wave instability analysis of thin liquid layer on a linearly heated plate. They showed the Marangoni effect on the magnitude of the critical Reynolds number. Miladinova et al.[7] extended the problem to the finite-amplitude long-wave instabilities of two-dimensional films. They derived a long-wave nonlinear evolution equation based on the Benney's approach[8] and confirmed the existence of permanent finite-amplitude waves of different kinds. Their linear stability analysis on the evolution equation also showed that the temperature decreasing along the plate gives an effect of stabilization on the film flow.

In this paper, our studies on the linear instability of a falling film along a linearly heated inclined plate will focus on the full-scale numerical dispersion relation other than the long-wave approximation due to its limitation demonstrated by many researchers (Refs. [9, 10]). In Section 1 the full-scale linear stability equations are derived and then an explicit dispersion relation under long-wave approximation are obtained. The numerical dispersion relations from full-scale linear stability equations are computed using Chebyshev spectral collocation method for both temporal and spatial modes (Sections 2 and 3 respectively). The effects of Weber numbers (We) and Marangoni numbers (Ma) on growth rate, marginal curve, critical Reynolds number, etc. are investigated.

1 Linear stability equations

We consider a thin liquid film with normal thickness d flowing down a linearly heated plate inclined at an angle β to the horizontal under gravitation as shown in Fig. 1. The ambient gas is assumed motionless with the constant temperature T_0, and the temperature distribution on the plate is $T_1 = T_0 + Ax$.

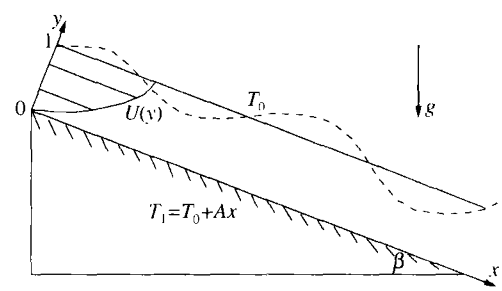

Fig. 1. Thin film flowing down a linearly heated inclined plate.

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Under the coordinates shown in Fig. 1, the dimensionless governing equations of the film flow are written as:

$$u_x + v_y = 0, \quad (1)$$

$$Re(u_t + uu_x + vu_y) = 2 - 2p_x + u_{xx} + u_{yy}, \quad (2)$$

$$Re(v_t + uv_x + vv_y) = -2\cot\beta - 2p_y + v_{xx} + v_{yy}, \quad (3)$$

$$Pr \cdot Re(\theta_t + u\theta_x + v\theta_y) = \theta_{xx} + \theta_{yy}. \quad (4)$$

The boundary conditions on the free surface $y = h(x, t)$ and the plate $y = 0$ are expressed respectively as

no-slip condition

$$u = v = 0, \quad y = 0, \quad (5)$$

dynamic condition

$$\boldsymbol{\Sigma} \cdot \boldsymbol{n} = -p_0\boldsymbol{n} + We_0(1 - Ca\theta)S\boldsymbol{n} - M\nabla_s\theta, \quad y = h(x,t), \quad (6)$$

kinematic condition

$$h_t + uh_x - v = 0, \quad y = h(x,t), \quad (7)$$

thermal conditon

$$\theta = x, \quad y = 0, \quad (8)$$

$$\nabla\theta \cdot \boldsymbol{n} + Bi\theta = 0, \quad y = h(x,t), \quad (9)$$

where $\boldsymbol{n} = \frac{(-h_x, 1)}{(1+h_x^2)^{\frac{1}{2}}}$, $\boldsymbol{t} = \frac{(1, h_x)}{(1+h_x^2)^{\frac{1}{2}}}$, $S = \frac{h_{xx}}{(1+h_0^2)^{\frac{3}{2}}}$ and $\boldsymbol{\Sigma} = \begin{pmatrix} u_x - p & \frac{1}{2}(u_y + v_x) \\ \frac{1}{2}(u_y + v_x) & v_y - p \end{pmatrix}$ are the normal unit vector, the tangential unit vector, the curvature of the surface and the stress tensor, respectively. The length, velocity, time and pressure are scaled with d, $U_0 = gd^2\sin\beta/2\nu$, d/U_0 and $\rho gd\sin\beta$ respectively, where ν is the kinematic viscosity and ρ is the density of the fluid; the temperature difference $T - T_0$ is scaled with Ad and the dimensionless temperature is denoted by $\theta = (T - T_0)/Ad$. The dimensionless parameters that appear in these equations are the Reynolds number $Re = gd^3\sin\beta/2\nu^2$, the Weber number $We_0 = \sigma_0/\rho gd^2\sin\beta$, the Capillary number $Ca = \gamma Ad/\sigma_0$, the Biot number $Bi = \alpha d/k$, the Marangoni number $Ma = \gamma Ad^2/\mu\kappa$ and the Prandtl number $Pr = \nu/\kappa$; where σ is the surface tension with the relation $\sigma(T) = \sigma_0(T_0) - \gamma(T - T_0)$, $\gamma = -\mathrm{d}\sigma/\mathrm{d}T$ is a positive constant for most common liquids, κ is the thermal diffusivity, k is the heat conductivity and α is the heat-transfer coefficient, respectively. Note that the relations of $M = Ma/(2Pr \cdot Re) = \gamma A/\rho gd\sin\beta$ and $Ca = M/We_0$ hold.

The Marangoni and Capillary numbers take positive or negative values depending on the direction of the temperature gradient. According to the analysis and treating of Miladinova et al.[7], the Capillary number is small enough to be ignored and the Weber number $We = We_0(1 - Ca\theta)$ is assumed constant.

When $Bi = 0$, there exists a basic steady solution of the film flow:

$$U(y) = 2(1-M)y - y^2, \quad (10)$$

$$P(y) = (1-y)\cot\beta + p_0, \quad (11)$$

$$\Theta(x,y) = x + \left[(1-M)\frac{y^3}{3} - \frac{y^4}{12}\right]Pr \cdot Re - \left(\frac{2}{3} - M\right)yPr \cdot Re. \quad (12)$$

It is easy to find that there does not exist the normal mode solution like $\phi(y)e^{i(kx-\omega t)}$ when $Bi \neq 0$, which is thus not considered in this paper. The physical significance of $Bi = 0$ is that the film surface is assumed insulated.

The disturbed flow can be decomposed into $u = U + u'$, $v = v'$, $p = P + p'$, $h = 1 + \eta$ and $\theta = \Theta + \theta'$. Further we introduce the disturbed stream function $\psi(y, x, t)$, $u' = \frac{\partial\psi}{\partial y}$, $v' = -\frac{\partial\psi}{\partial x}$ and assume that there are the normal mode solutions of the form:

$$\psi(y,x,t) = f(y)e^{i(kx-\omega t)},$$

$$\theta(y,x,t) = g(y)e^{i(kx-\omega t)}. \quad (13)$$

Substituting them into the small disturbed linearized equations and the corresponding boundary conditions, we can eliminate the disturbed pressure p' and free surface thickness η; finally we obtain the linear stability equations:

$$f'''' - 2k^2f'' + k^4f = ikRe\left[\left(U - \frac{\omega}{k}\right)(f'' - k^2f) - U''f\right], \quad (14)$$

$$g'' - k^2g = ikPr \cdot Re\left[\left(U - \frac{\omega}{k}\right)g - D\Theta \cdot f\right] + Pr \cdot Ref' \quad (15)$$

and linear boundary conditions:

$$f(0) = 0, \quad f'(0) = 0, \quad (16)$$

$$f''(1) + \left(k^2 + \frac{2k}{kU(1) - \omega}\right)f(1) + 2ikMg(1) = 0, \quad (17)$$

$$f'''(1) - [3k^2 + iRe(kU(1) - \omega)]f'(1) - 2ikReMf(1) + \frac{2i(k^2\cot\beta + 2ik^3M + k^4We)}{kU(1) - \omega}f(1) = 0, \quad (18)$$

$$g(0) = 0, \quad (19)$$

$$g'(1) - [Pr \cdot Re(1-2M) - 2ik]\frac{k}{kU(1)-\omega}f(1) = 0. \tag{20}$$

So far, for the first time we establish the linear stability equations for the linearly heated film flowing down an inclined plate. They are ordinary differential equations in terms of f, g as a two-point boundary value problem. If there exists a non-trivial solution for the equations, a corresponding dispersion relation $D(k, \omega; Re, We, \beta, Pr, Ma) = 0$ should be satisfied, and we need to solve an eigenvalue problem. Because it is impossible to find the explicit analytical dispersion relation, the dispersion relation has to be obtained numerically. In our calculations the Chebyshev collocation method[11] is used, and the QZ algorithm[12] is utilized to solve the resulting general eigenvalue problem. In order to check the validation of the equations and computational codes, we proceed twofold as follows: first, our results are in accurate agreement compared with those from Refs. [9, 10] for non-heated plate; secondly, the same explicit dispersion relation[7] under the long-wave approximation is derived using expansion of small parameter k in Eqs. (14)~(20). We set

$$f = f_0 + kf_1 + \cdots, \quad g = g_0 + kg_1 + \cdots,$$
$$c = c_0 + kc_1 + \cdots.$$

Substitute them into Eqs. (14) ~ (20) to obtain the k-zero and first order approximation. Finally the dispersion relation gives

$$c = c_0 + kc_1 = \left(2 - \frac{Ma}{PrRe}\right) + kc_1,$$
$$c_1 = \frac{i}{Re}\left\{\frac{8}{15}Re^2 - \frac{2}{3}Re \cdot \cot\beta - \frac{2}{3}k^2 ReWe + \frac{5}{6}\left(Pr - \frac{8}{25}\right)\frac{ReMa}{Pr} - \frac{1}{2}\frac{Ma^2}{Pr}\right\}. \tag{21}$$

Because our dimensionless groups defined above differ from those in Ref. [7], it is easier to convert with each other and prove that the present dispersion relation is the same as that in [7].

2 Temporal mode

Fig. 2 shows the temporal growth rate computed from full-scale stability equations (14) ~ (20) versus different We, Ma and Re. From Fig. 2(a) we can see that the increase in surface tension has stabilization effect on the film flow, which makes the maximum growth rate and the corresponding cutoff wavenumbers decrease. However, the Weber number almost has no influence on growth rate as $k_r \to 0$. Because $Ma > 0$ implies that the plate temperature is linearly increased, otherwise linearly decreased for most common liquids $\gamma > 0$ from the definition of the Marangoni number. Fig. 2(b) shows that the positive Marangoni number has destabilization effect, otherwise stabilization; but the wavenumbers corresponding to the maximum growth rate are hardly changed. The curves at the right end are already in the stable

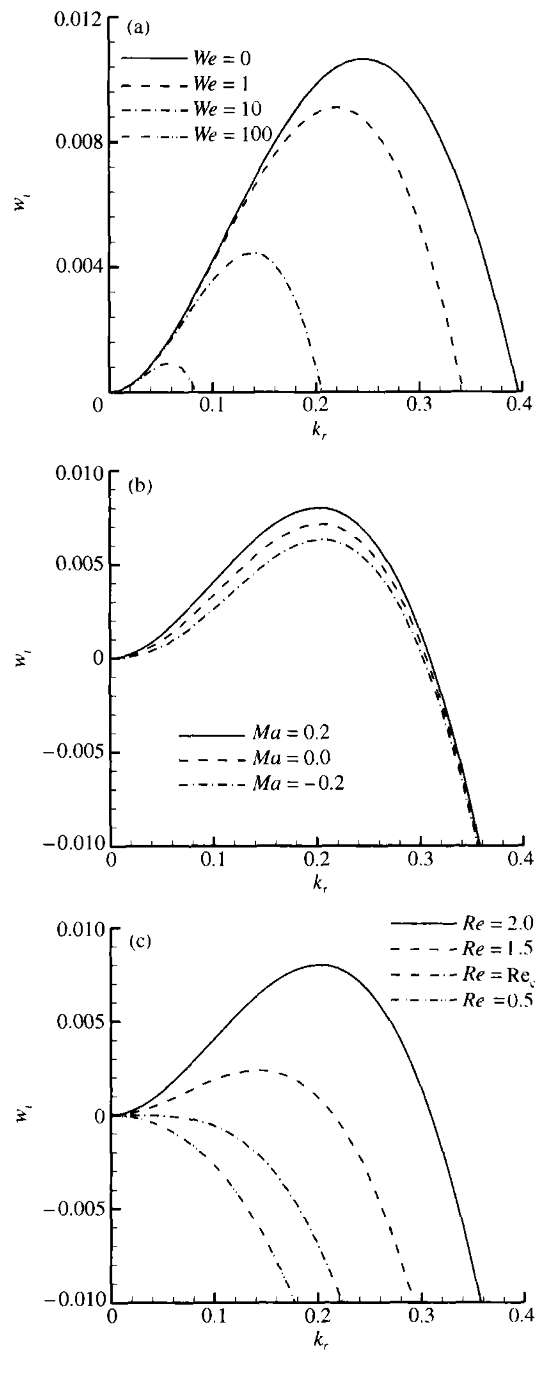

Fig. 2. Temporal growth rate versus wavenumber. (a) Effect of Weber number ($Re = 2$, $\beta = 45°$, $Pr = 10$, $Ma = 0.2$), (b) effect of Marangoni number ($Re = 2$, $We = 2$, $\beta = 45°$, $Pr = 10$), (c) effect of Reynolds number ($We = 2$, $\beta = 45°$, $Pr = 10$, $Ma = 0.2$).

range, and the cutoff wavenumber which has been in the region beyond limit of long wave approximation has slightly increased as Marangoni number increases. The effect of viscosity on the film instability can be seen easily from Fig. 2 (c). Below the critical Reynolds number, the film flow becomes stable.

As mentioned above, the limitation of the long wave approximation is pointed out by some authors[9,10]. For example, although there exist absolutely instability regions according to the analysis of the long wave equation for the isothermal film flow, in fact, the experimental and the numerical results from the full-scale dispersion relation show that the film flow is only convectively unstable. For comparison, we draw the temporal growth rate given by full-scale equations (14) ~ (20) and from the long wave approximation (21) in Fig. 3. It is seen from Fig. 3 (a) that the long wave approximation is correct only about $k_r \leqslant 0.05$, and it cannot predict the critical wavenumber. Fig. 3 (b) is drawn with the small Reynolds number and the large Weber number; the cutoff wavenumber is within the long wave region in this case, both the curves are closer to each other and the long wave approximation is suitable.

Fig. 4 shows the effect of Marangoni number on the marginal stability with different Weber numbers. Generally the increase in Marangoni number makes the film flow more unstable. For most instances the critical Reynolds number occurs at the zero wavenumber. It can be noticed that, however, the critical Reynolds number can occur in the region of wavenumber $k = (0.1 \sim 0.2)$ for small Weber numbers and negative Ma numbers, which is beyond the long wave region. Therefore we should be careful in determining the critical Reynolds number by the long-wave equation. In addition, Fig. 4 shows that when the Reynolds number is far away from the critical, the Marangoni number has only a weak effect on the marginal wavenumber.

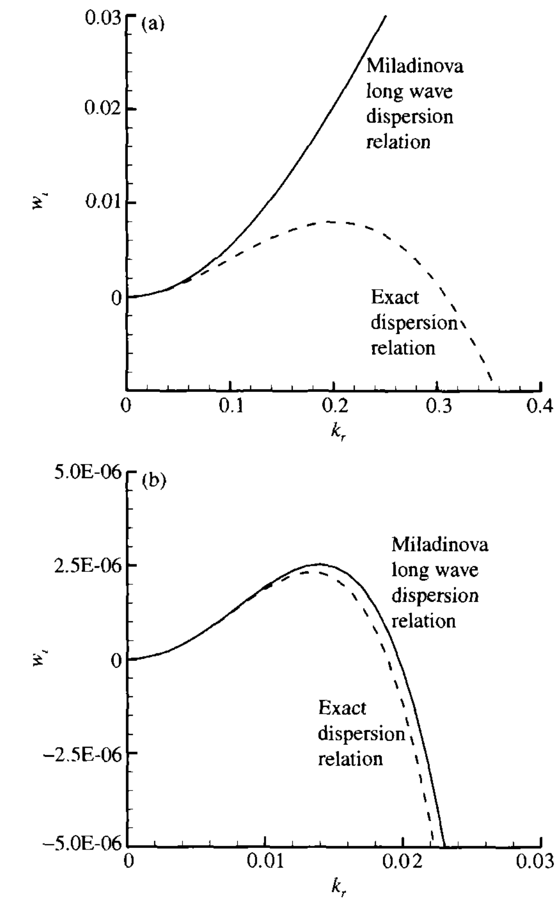

Fig. 3. Temporal growth rate versus wavenumber (When $\beta = 45°$, $Pr = 10$, $Ma = 0.2$). (a) $Re = 2$, $We = 2$; (b) $Re = 1$, $We = 100$. The solid line for long-wave approximation, the dashed line for full-scale dispersion relation.

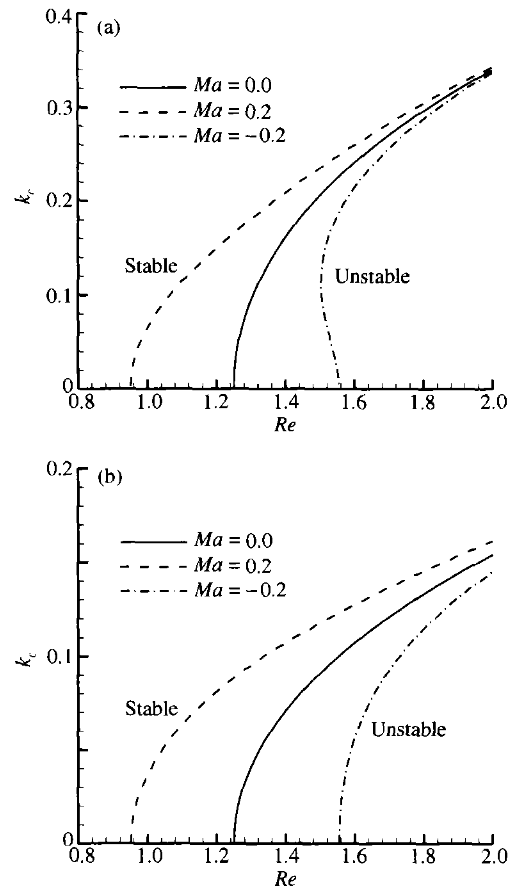

Fig. 4. The effect of Weber number and Marangoni number on the marginal curves ($\beta = 45°$, $Pr = 10$). (a) $We = 1$, (b) $We = 20$.

3 Spatial mode

For convectively unstable film flow, we can further study the properties of the spatially amplifying waves. Considering spatial mode, we set ω real number and k complex number, $k = k_r + \mathrm{i}k_i$. From Eqs. (14) ~ (20), we can obtain the spatially amplifying wave ($\sim e^{-k_i x}$). Fig. 5 is such an example. Gaster[13] has proved that there exists a conversion relation between the spatial and the temporal growth rates when they are both small:

$$-k_i(S) = \frac{\omega_i(T)}{\partial \omega_r / \partial k_r},$$

where S and T mean the values computed from spatial and temporal modes respectively. The results from the above Gaster's transformation are the same as those obtained from directly numerical computation of Eqs. (14) ~ (20). It is proved again that Eqs. (14) ~ (20) and codes are reliable.

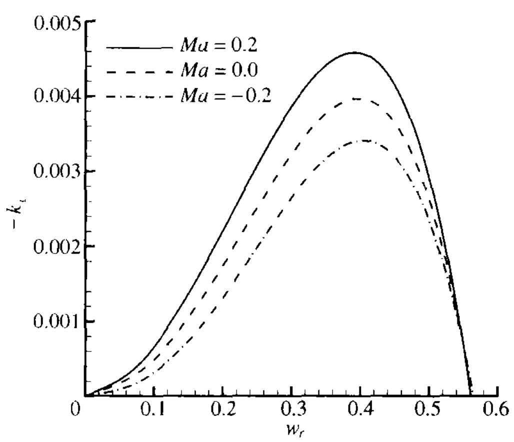

Fig. 5. Spatial growth rate versus the frequency ($Re = 2$, $We = 2$, $\beta = 45°$, $Pr = 10$).

4 Conclusion

The linear instability of a liquid film flowing down a linearly heated inclined plate is investigated in the present paper. Under assumption of adiabatic free surface and small Marangoni number the full-scale linear stability equations with normal mode decomposition are derived from the N-S equations, and an explicit dispersion relation under the long-wave approximation, which is in exact agreement with Miladinova's, is yielded. The numerical dispersion relations from full-scale linear stability equations are computed using Chebyshev spectral collocation method for both temporal and spatial modes. The effects of We and Ma numbers on growth rate, marginal curve, critical Reynolds number, etc. are considered. We found that the increase in Marangoni number has destabilization effect on the surface wave instability. Although the long-wave approximation is available at most situations for thin film flows, the critical Reynolds number determined by the long-wave approximation is problematic compared with exact dispersion relation in certain parameter regions. Hence, it must be careful to use long wavenumber approximation. For the spatial amplifying waves, we obtained the same conclusions as those of temporal modes.

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